Quantization conditions in Bogomolny's transfer operator method

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Bogomolny's transfer operator method plays a significant role in the study of quantum chaos, along with other well known methods like Gutzwiller's trace formula and the dynamical zeta function, which generalize the Einstein-Brillouin-Keller quantization rule from integrable systems to chaotic systems. According to the theory, the Fredholm determinant of the transfer operator, defined on a Poincaré section of a classical physical system, provides a quantization condition to the energy spectrum of the corresponding quantum system. This study presents two factorization formulas, which relate different quantization conditions defined on different classical trajectory segments. These explicit relations answer the question of why all these classical quantization conditions determine exactly the same energy spectrum of the corresponding quantum systems. As an example, these formulas are illustrated in the equilateral triangular billiard.

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I. INTRODUCTION

A way to generalize the Wentzel-Kramers-Brillouin and the Einstein-Brillouin-Keller (EBK) quantization rules from integrable systems to chaotic (nonintegrable) systems had been sought for a long time [1]. In the 1950s, without the motivation of quantum chaos, Selberg derived his famous trace formula [2]. This formula describes the discrete quantum energy spectrum of a free particle on the modular surface on which the dynamics of the particle's motion is strongly chaotic [1]. Shortly thereafter, Sinai and Smale [3] realized that this formula could be interpreted as a sum over all periodic orbits of the particle motion on the surface. Since then, the relation between the energies of a quantum system and its classical chaotic dynamics has become much clearer [1]. This relation is essentially involved in many newly developed quantization rules in the last decades. Examples include the transfer operator method due to Ruelle [4] on the modular surface [5–7] and semiclassical methods for general physical systems, such as Gutzwiller's trace formula [8] and the functional determinant [9]. A common point of all these methods is that the quantization conditions are established from the periodic orbits of the dynamical systems.

Another quantization method based on classical dynamics is the transfer operator method due to Bogomolny [10]. Unlike other methods, the quantization condition in this method is not based on periodic orbits but only on segments of certain trajectories in classical dynamics. The freedom of trajectory choice is such that the quantization conditions in this method are not unique and, in fact, infinitely many different conditions can exist. From a classical perspective, one might speculate how all these different quantization conditions, defined on independent classical trajectories, determine exactly the same energy spectrum. This work provides a simple argument with exact formulas (19) and (20) to answer this question. Finally, a numerical test that confirms these formulas on the equilateral triangular quantum billiard is discussed.

II. TRANSFER OPERATOR METHOD

Bogomolny's transfer operator method is briefly introduced here. Consider a particle with energy E, moving in some k-dimensional physical system, (for examples of various systems, see Refs. [11–19]). Select a Poincaré section (PS) Σ in the configuration space of this system, such that almost all classical trajectories pass this section. The transfer operator $\mathcal{T}(E)$ is defined semiclassically as the integral operator [10]

$$\mathcal{T}(E)\psi(q)=\int_{\Sigma}T(q,q',E)\psi(q')dq', \tag{1}$$

acting on some function $\psi(q')$ on Σ. The integral kernel

$$T(q,q',E)=\sum_{\text{class traj}}\frac{1}{(2\pi i\hbar)^{(k-1)/2}}\sqrt{\left|\det\frac{\partial^2 S(q,q',E)}{\partial q\,\partial q'}\right|}\times\exp[iS(q,q',E)/\hbar-i\nu\pi/2] \tag{2}$$

is defined as the sum over all possible classical trajectories from the initial point $q'\in\Sigma$ to the final point $q\in\Sigma$ in the configuration space at energy E. These trajectories cannot have other crossing points through Σ between q' and q, in the sense that they cannot pass Σ in the same direction as they pass q'. For example, selecting Σ^{I} in Fig. 1(a) as a PS, the next crossing point after the initial point q_1 is q_3, not q'. The function $S(q,q',E)$ in Eq. (2) is the action of the trajectory from q' to q at energy E and the Maslov index ν is related to the number of points, at which semiclassical approximation is not valid [10]. For a free particle that moves

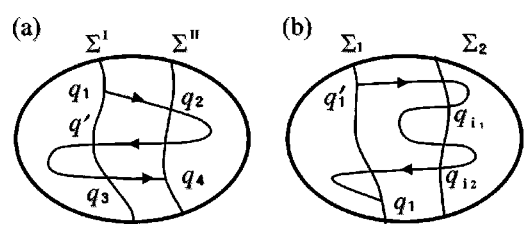

FIG. 1. (a) On the PS Σ^{I}, the next crossing point after q_1 is q_3 (not q'). Moreover, the transfer operators on Σ^{I} and Σ^{II} can be related by Green functions in Eq. (4). (b) The trajectory of the kernel $T^3(q_1,q_1')$ in Eq. (10) passes through q_1', q_{i_1}, q_{i_2}, and q_1.

in a two-dimensional billiard system, the Maslov index is double the number of reflections of the trajectory at the billiard boundary [10]. The partial derivative $\partial^2 S(q,q',E)/\partial q \partial q'$ in Eq. (2) is a square matrix of dimension 2 for $k=3$ and a scalar for $k=2$, where the notation "det" before the derivative can be omitted, e.g., Eq. (22). In this approximation, the kernel (2) is unitary and satisfies the relation

$$\int_{\Sigma} T(q,q'',E)T(q'',q',E)dq'' = T^2(q,q',E),$$

where the connection of the trajectories from q' to q'' and from q'' to q must be smooth at q''; that is, the direction of the incoming trajectory at q'' and the direction of the outgoing trajectory at q'' are the same. Moreover, $T(q,q',E)$ vanishes if no trajectory that fulfills this condition exists from q' to q at energy E. According to Bogomolny's theory [10], in the semiclassical limit $\hbar \rightarrow 0$, the zeros of the Fredholm determinant

$$\det[1-\mathcal{T}(E)]=0, \tag{3}$$

of the transfer operator $\mathcal{T}(E)$ are the energies of the corresponding quantum system yielding a quantization condition.

The choice of the PS is arbitrary, as long as almost all trajectories run through this section. Given two sections Σ^{I} and Σ^{II}, Ref. [10] argues that the kernels T^{I} and T^{II} of the corresponding transfer operators $\mathcal{T}^{\mathrm{I}}$ and $\mathcal{T}^{\mathrm{II}}$ are related by

$$T^{\mathrm{II}}(q_4,q_2,E) = \int_{\Sigma^{\mathrm{I}}}\int_{\Sigma^{\mathrm{I}}} G_{43} T^{\mathrm{I}}(q_3,q_1,E) G^*_{21} dq_3 dq_1, \tag{4}$$

in which $G_{ij}=G(q_i,q_j,E)$ is the Green function corresponding to the transition from a point q_j on Σ^{I} to a point q_i on Σ^{II} at energy E without other crossing points through Σ^{I} or Σ^{II} in between [Fig. 1(a)], where G^*_{ij} represents the complex conjugate of G_{ij}. Since the Green function is a unitary transformation [10],

$$\int_{\Sigma} G(q',q'',E)G(q'',q,E)^* dq'' = \delta(q'-q), \tag{5}$$

$\mathcal{T}^{\mathrm{I}}$ and $\mathcal{T}^{\mathrm{II}}$ are similar operators under different bases, and the corresponding determinants in Eq. (3) are identical.

This argument is valid only when the Green functions in Eq. (4) can be found. However, this condition is not met in general. For example, a trajectory from q_1 could have crossed Σ^{I} many times before arriving at q_2 if Σ^{I} and Σ^{II} are not close enough. Accordingly, the Green function G^*_{21} in Eq. (4), defined on classical trajectories, has more than one crossing point on Σ^{I}, which contradicts the requirement on G_{ij}. Thus, the transfer operators on the new and old sections are no longer similar operators and their Fredholm determinants are in general different functions. From the quantum mechanical perspective, these functions should clearly have the same zeros, because these zeros correspond to a unique energy spectrum. However, from the classical perspective, the reason why all the zeros of different functions (3) constructed on different PS's should be exactly identical, is nontrivial. This question can be clarified using the following factorization formula.

III. DETERMINANT FACTORIZATION

Consider first a transfer operator $\mathcal{T}^{\mathrm{II}}$ defined on some PS $\Sigma^{\mathrm{II}}=\Sigma_1 \cup \Sigma_2$, composed of two connected, but nonoverlapping, subsections Σ_1 and Σ_2. The operator (1), split into two parts,

$$\mathcal{T}^{\mathrm{II}}\psi(q) = \int_{\Sigma_1} T^{\mathrm{II}}(q,q_1')\psi(q_1')dq_1' + \int_{\Sigma_2} T^{\mathrm{II}}(q,q_2')\psi(q_2')dq_2', \tag{6}$$

can be expressed as

$$\mathcal{T}^{\mathrm{II}}\begin{pmatrix}\psi(q_1)\\ \psi(q_2)\end{pmatrix} = \begin{pmatrix}\mathcal{T}_{11} & \mathcal{T}_{12}\\ \mathcal{T}_{21} & \mathcal{T}_{22}\end{pmatrix}\begin{pmatrix}\psi(q_1)\\ \psi(q_2)\end{pmatrix}, \tag{7}$$

with $q_1,q_1' \in \Sigma_1$, $q_2,q_2' \in \Sigma_2$, and

$$\mathcal{T}_{ij}\psi(q_j) \coloneqq \int_{\Sigma_j} T^{\mathrm{II}}(q_i,q_j')\psi(q_j')dq_j', \quad i,j \in \{1,2\}, \tag{8}$$

where $\psi(q_j)$ is some function defined on Σ_j. Notably, the integral in Eq. (8) is conventionally defined as an integral operator by using the notation $\mathcal{T}_{ij}\psi(q_i)=[\mathcal{T}_{ij}\psi](q_i)$, indicating that $[\mathcal{T}_{ij}\psi]$ is a function of q_i. The modified notation $\mathcal{T}_{ij}\psi(q_j)$ in Eq. (8) does not refer to a function of q_j. Instead, this notation makes it possible to express Eq. (6) as the matrix equation Eq. (7) and shorten many equations in the following discussion. As in Eq. (2), the kernel $T^{\mathrm{II}}(q_i,q_j')$ in Eq. (8) consists of trajectories from q_j' to q_i without crossing Σ_1 and Σ_2 in between, in the sense of the crossing point defined above.

Next, $\Sigma^{\mathrm{I}}=\Sigma_1$ is selected as a new PS with the corresponding transfer operator $\mathcal{T}^{\mathrm{I}}$ and its kernel $T^{\mathrm{I}}(q_1,q_1')$. The aim is to determine the relation between kernels $T^{\mathrm{I}}(q_1,q_1')$ and $T^{\mathrm{II}}(q_1,q_1')$. Accordingly, $T^{\mathrm{I}}(q_1,q_1')$ is written as a sum over all trajectories from q_1' to q_1,

$$T^{\mathrm{I}}(q_1,q_1') = \sum_{n=1}^{\infty} T^n(q_1,q_1'), \tag{9}$$

where the first term $T(q_1,q_1')$ in the sum is the kernel $T^{\mathrm{II}}(q_1,q_1')$ of $\mathcal{T}_{11}$ in Eq. (8), which corresponds to trajectories from q_1' to q_1 without crossing Σ_2. For $1<n$, the composition kernel

$$T^n(q_1,q_1') \coloneqq \int_{\Sigma_2} T^{\mathrm{II}}(q_1,q_{i_{n-1}})T^{\mathrm{II}}(q_{i_{n-1}},q_{i_{n-2}}) \dots \times T^{\mathrm{II}}(q_{i_2},q_{i_1})T^{\mathrm{II}}(q_{i_1},q_1')dq_{i_{n-1}} \dots dq_{i_1} \tag{10}$$

corresponds to the trajectories that cross the section Σ_2 $n-1$ times at $q_{i_l} \in \Sigma_2$ with $l=1, \dots, n-1$. Figure 1(b) shows an example of $T^3(q_1,q_1')$ with a trajectory that crosses Σ_2 twice. Consequently, $T^{\mathrm{I}}(q_1,q_1')$ in Eq. (9) is related to $T^{\mathrm{II}}(q_1,q_1')$ in Eq. (8) by

$$T^{\mathrm{II}}(q_1,q_1')+\sum_{n=2}^{\infty} T^n(q_1,q_1')=T^{\mathrm{I}}(q_1,q_1'). \tag{11}$$

Defining a new operator $\mathcal{A}$ with the kernel $\Sigma_{n=1}^{\infty}T^n(q_1,q_2')$, by replacing q_1' in Eq. (10) by q_2', yields the identity,

$$\sum_{n=1}^{\infty} T^n(q_1,q_2')-\sum_{n=2}^{\infty} T^n(q_1,q_2')=T^{\mathrm{II}}(q_1,q_2'). \tag{12}$$

The kernels of $\mathcal{A}\mathcal{T}_{21}$ and $\mathcal{A}\mathcal{T}_{22}$ are then, respectively,

$$\int_{\Sigma_2}\sum_{n=1}^{\infty} T^n(q_1,q_2'')T_{21}(q_2'',q_1')dq_2''=\sum_{n=2}^{\infty} T^n(q_1,q_1'), \tag{13}$$

$$\int_{\Sigma_2}\sum_{n=1}^{\infty} T^n(q_1,q_2'')T_{22}(q_2'',q_2')dq_2''=\sum_{n=2}^{\infty} T^n(q_1,q_2'), \tag{14}$$

which consist of trajectories from q_1', respectively, q_2' to q_1, crossing Σ_2 at least once. Using the definition of $\mathcal{A}\mathcal{T}_{21}$ and $\mathcal{A}\mathcal{T}_{22}$ in Eqs. (13) and (14), kernel equations (11) and (12) yield the operator equations

$$\mathcal{T}_{11}+\mathcal{A}\mathcal{T}_{21}=\mathcal{T}^{\mathrm{I}} \quad \text{and} \quad \mathcal{A}(1-\mathcal{T}_{22})=\mathcal{T}_{12}. \tag{15}$$

The necessary and sufficient condition for $\mathcal{T}^{\mathrm{II}}(E)$ to have an eigenvalue 1 at some E is

$$\det[1-\mathcal{T}^{\mathrm{II}}(E)]=0, \tag{16}$$

where, according to Eq. (7), one has

$$[1-\mathcal{T}^{\mathrm{II}}(E)]\psi(q)=\begin{pmatrix} 1-\mathcal{T}_{11} & -\mathcal{T}_{12} \\ -\mathcal{T}_{21} & 1-\mathcal{T}_{22}\end{pmatrix}\begin{pmatrix}\psi(q_1)\\ \psi(q_2)\end{pmatrix}. \tag{17}$$

Condition (16) is identical to applying an operator with unit determinant from the left-hand side of Eq. (17) and then taking its determinant,

$$\det\left[\begin{pmatrix}1 & \mathcal{A}\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1-\mathcal{T}_{11} & -\mathcal{T}_{12} \\ -\mathcal{T}_{21} & 1-\mathcal{T}_{22}\end{pmatrix}\right]=\det\begin{pmatrix} 1-\mathcal{T}^{\mathrm{I}} & 0 \\ -\mathcal{T}_{21} & 1-\mathcal{T}_{22}\end{pmatrix}=0, \tag{18}$$

where the two identities in Eq. (15) have been used. Consequently, to find an eigenvalue 1 for $\mathcal{T}^{\mathrm{II}}(E)$ is equivalent to finding the zeros of the determinant

$$\det[1-\mathcal{T}^{\mathrm{II}}(E)]=\det[1-\mathcal{T}^{\mathrm{I}}(E)]g(E), \tag{19}$$

where $g(E)=\det[1-\mathcal{T}_{22}(E)]$ is a bounded function, such that $|g(E)|<\infty$, because $\det[1-\mathcal{T}(E)]$ in Bogomolny's method is not singular.

This factorization formula relates the quantization condition $\det[1-\mathcal{T}^{\mathrm{I}}(E)]$ on Σ^{I} to the quantization condition $\det[1-\mathcal{T}^{\mathrm{II}}(E)]$ on Σ^{II}. Notably, $\mathcal{T}^{\mathrm{I}}$ and $\mathcal{T}^{\mathrm{II}}$ are the transfer operators of $\Sigma^{\mathrm{I}}=\Sigma_1$ and $\Sigma^{\mathrm{II}}=\Sigma_1\cup\Sigma_2$, respectively. However, $\mathcal{T}_{22}$ is not the transfer operator of Σ_2, because $\mathcal{T}_{22}$ does not include all trajectories from $q_2'\in\Sigma_2$ to $q_2\in\Sigma_2$, but only those from q_2' to q_2 that do not cross Σ_1. Moreover, $\mathcal{T}_{12}$ is nonzero because not all trajectories starting from Σ_2 come back to Σ_2 without crossing Σ_1 in between, unless Σ_1 and Σ_2 are in different parts of a nonconnected system. Together with the second equality of Eq. (15), it implies that $\mathcal{T}_{22}$ does not have eigenvalue 1 and $g(E)$ is nonzero. Therefore, all zeros of $\det[1-\mathcal{T}^{\mathrm{II}}(E)]$ in Eq. (19) arise from $\det[1-\mathcal{T}^{\mathrm{I}}(E)]$ and vice versa. Consequently, the quantization conditions on different PS's determine exactly the same energy spectrum. Furthermore, $\det[1-\mathcal{T}^{II}(E)]$ and $\det[1-\mathcal{T}^{\mathrm{I}}(E)]$ are identical only if $g(E)$ is unity. In this case, $\mathcal{T}_{22}(E)$ vanishes and no trajectory connects two points on Σ_2 without crossing Σ_1, an example of which can be found in Eq. (27).

In the general case in which Σ^{I} and Σ^{II} are not connected, a section Σ' connecting Σ^{I} and Σ^{II} can be chosen such that $\Sigma=\Sigma^{\mathrm{I}}\cup\Sigma'\cup\Sigma^{\mathrm{II}}$ is connected. Applying Eq. (19), the determinant for Σ can then be factorized as

$$\begin{aligned}\det[1-\mathcal{T}(E)]&=\det[1-\mathcal{T}^{\mathrm{I}}(E)]g^{I}(E)\\ &=\det[1-\mathcal{T}^{\mathrm{II}}(E)]g^{\mathrm{II}}(E),\end{aligned}$$

with certain nonzero functions $g^{\mathrm{I}}(E)$ and $g^{\mathrm{II}}(E)$. Finally a general formula that relates various quantization conditions on arbitrary PS's is obtained,

$$\det[1-\mathcal{T}^{\mathrm{II}}(E)]=\det[1-\mathcal{T}^{\mathrm{I}}(E)]G(E), \tag{20}$$

with the nonzero bounded function $G(E)=g^{\mathrm{I}}(E)/g^{\mathrm{II}}(E)$. Notably, Bogomolny's quantization condition (3) holds under the semiclassical assumption. However, the derivation from Eq. (3) to formulas (19) and (20) is only a reformulation of Eq. (3) without extra approximations besides the semiclassical approximation.

IV. EQUILATERAL TRIANGULAR BILLIARD

A. Transfer operators for the classical billiard

As an example of Eq. (19), consider an equilateral triangular billiard B bounded by three sides $\Sigma_1=\overline{p_1p_2}$, $\Sigma_2=\overline{p_2p_3}$, and $\Sigma_3=\overline{p_3p_1}$ with corners $p_1(0,0)$, $p_2(-1/2,\sqrt{3}/2)$, and $p_3(1/2,\sqrt{3}/2)$, as shown in Fig. 2(a). Let $\Sigma^{\mathrm{I}}=\Sigma_1$ be a PS and $s\in[0,1]$ be the local coordinate on Σ^{I}; that is, s represents the distance from the origin p_1 to a point $q(s)\in\Sigma^{\mathrm{I}}$. In Cartesian coordinates, the point $q(s)$ is at $q(-s/2,\sqrt{3}s/2)$.

The first step in defining the transfer operator on Σ^{I} is to determine all trajectories from $q'(s')\in\Sigma^{\mathrm{I}}$ to $q(s)\in\Sigma^{\mathrm{I}}$. The question is equivalent to asking how many images of an object at position q an observer at position q' can detect if the observer stands inside triangle B in Fig. 2(a), with two mirror walls Σ_2 and Σ_3. Five images are at $q=q_n$ with $n=1,2,\ldots,5$ on five sides of the hexagon in Fig. 2(a). The distances $\overline{q_nq'}$ to q' equal the lengths l_n of the five trajectories from q' to q in B. For example, the trajectory from q'

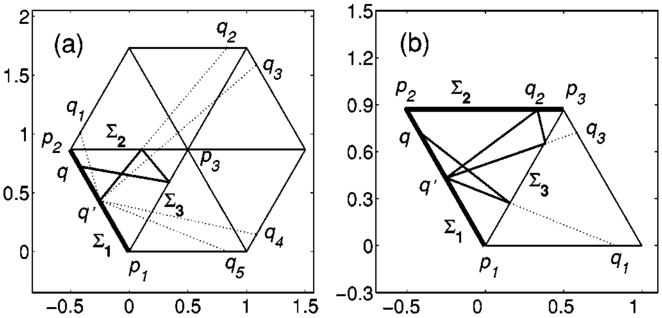

FIG. 2. (a) For Σ^{I}, five trajectories (dashed lines) from q' to five images q_n of q, with length unit L in Eq. (25). (b) For Σ^{II}, one trajectory from q' on Σ_1 to q on Σ_1 and two trajectories from q' on Σ_1 to q on Σ_2.

to q plotted as a solid line in Fig. 2(a), has the same length l_2 as $\overline{q_2 q'}$. The Maslov index ν for this trajectory is 6, because, for a free particle in a billiard system, ν is double the number of reflections at the billiard boundary [10]. Table I lists the locations of q_n and their Maslov indices ν. Thus, the length l_n, the action $S(q_n,q',E)=\sqrt{2\mu E}l_n$, where μ is the mass of the particle, and $\partial^2 S(q_n,q',E)/\partial q_n \partial q'$ of all trajectories, can be determined. Taking all these quantities into account, the kernel of the transfer operator for the PS Σ^{I} is

$$T^{\mathrm{I}}(q,q')=k_{11}(q,q'), \tag{21}$$

where $k_{11}(q,q')$ is the special case $\alpha=\beta=1$ of the general form

TABLE I. The initial points q' on Σ_j, the images q_n of final point q on Σ_i, and the Maslov indices ν of various kinds of trajectories through Poincaré sections Σ^{I}, Σ^{II}, and Σ^{III} in Fig. 2.

PS	$q'(s')\in$	Σ_j	Images $q_n(s)$ of	$q(s)\in\Sigma_i$	ν
Σ^{I}	$\left(\frac{-s'}{2},\frac{\sqrt{3}s'}{2}\right)$	Σ_1	$q_1\left(\frac{-s}{2},\frac{2\sqrt{3}-\sqrt{3}s}{2}\right)$	Σ_1	4
			$q_2(s,\sqrt{3})$	Σ_1	6
			$q_3\left(\frac{3-s}{2},\frac{\sqrt{3}+\sqrt{3}s}{2}\right)$	Σ_1	8
			$q_4\left(\frac{3-s}{2},\frac{\sqrt{3}-\sqrt{3}s}{2}\right)$	Σ_1	6
			$q_5(s,0)$	Σ_1	4
Σ^{II}	$\left(\frac{-s'}{2},\frac{\sqrt{3}s'}{2}\right)$	Σ_1	$q_1(s,0)$	Σ_1	4
			$q_2\left(\frac{2s-3}{2},\frac{\sqrt{3}}{2}\right)$	Σ_2	2
			$q_3\left(\frac{3-s}{2},\frac{\sqrt{3}(s-1)}{2}\right)$	Σ_2	4
	$\left(\frac{2s'-3}{2},\frac{\sqrt{3}}{2}\right)$	Σ_2	$q_4\left(\frac{-s}{2},\frac{\sqrt{3}s}{2}\right)$	Σ_1	2
			$q_5(s,0)$	Σ_1	4
			$q_6\left(\frac{3-s}{2},\frac{\sqrt{3}(s-1)}{2}\right)$	Σ_2	4
Σ^{III}	$\left(\frac{-s'}{2},\frac{\sqrt{3}s'}{2}\right)$	Σ_1	$q_1\left(\frac{2s-3}{2},\frac{\sqrt{3}}{2}\right)$	Σ_2	2
			$q_2\left(\frac{3-s}{2},\frac{\sqrt{3}(3-s)}{2}\right)$	Σ_3	2
	$\left(\frac{2s'-3}{2},\frac{\sqrt{3}}{2}\right)$	Σ_2	$q_3\left(\frac{-s}{2},\frac{\sqrt{3}s}{2}\right)$	Σ_1	2
			$q_4\left(\frac{3-s}{2},\frac{\sqrt{3}(3-s)}{2}\right)$	Σ_3	2
	$\left(\frac{3-s'}{2},\frac{\sqrt{3}(3-s')}{2}\right)$	Σ_3	$q_5\left(\frac{-s}{2},\frac{\sqrt{3}s}{2}\right)$	Σ_1	2
			$q_6\left(\frac{2s-3}{2},\frac{\sqrt{3}}{2}\right)$	Σ_2	2

$$k_{\alpha\beta}(q,q')=\sum_{\text{class traj}}\frac{\sqrt{\left|\frac{\partial^2 S(q,q',E)}{\partial q\,\partial q'}\right|}}{(2\pi i\hbar)^{1/2}}\times\exp[iS(q,q',E)/\hbar]-(i\nu\pi/2), \qquad (22)$$

where the sum runs over all trajectories from $q'\in\Sigma_\beta$ to $q\in\Sigma_\alpha$ with $\alpha,\beta\in\{1,2,3\}$.

For the second PS, select $\Sigma^{\mathrm{II}}=\Sigma_1\cup\Sigma_2$. As in the previous case, determining all trajectories from $q'(s')\in\Sigma^{\mathrm{II}}$ to $q(s)\in\Sigma^{\mathrm{II}}$ with $s\in[0,2]$ on Σ^{II}, is equivalent to determining all images of an object at q in the triangle in Fig. 2(b) with a mirror wall Σ_3. The corresponding transfer operator has the kernel,

$$T^{\mathrm{II}}(q,q')=\sum_{\alpha,\beta\in\{1,2\}}k_{\alpha\beta}(q,q')\chi_\alpha(q)\chi_\beta(q') \qquad (23)$$

with

$$\chi_\alpha(q)=\begin{cases}1 & \text{for } q\in\Sigma_\alpha\\ 0 & \text{for } q\notin\Sigma_\alpha.\end{cases}$$

For $q,q'\in\Sigma_1$, $T^{\mathrm{II}}(q,q')$ consists of only one trajectory, which is reflected by Σ_3, and is as long as $\overline{q_1q'}$ in Fig. 2(b). For $q=q_2\in\Sigma_2$ and $q'\in\Sigma_1$, $T^{\mathrm{II}}(q,q')$ is comprised of two trajectories that are as long as $\overline{q_2q'}$ and $\overline{q_3q'}$. Table I lists all these and other trajectories for Σ^{II}.

The following section is the most conventional choice of PS for a billiard system, namely, the whole boundary. For the billiard considered here, this section equals $\Sigma^{\mathrm{III}}=\Sigma_1\cup\Sigma_2\cup\Sigma_3$. By a similar argument as for Σ^{II}, the kernel of the transfer operator for Σ^{III} is

$$T^{\mathrm{III}}(q,q')=\sum_{\alpha,\beta\in\{1,2,3\}}k_{\alpha\beta}(q,q')\chi_\alpha(q)\chi_\beta(q'), \qquad (24)$$

to which six trajectories that connect points q' and q_n and are given in Table I contribute.

B. Semiclassical quantization

The exact energies of the equilateral triangular quantum billiard are

$$E=\frac{8\hbar^2\pi^2(n^2+m^2-mn)}{9\mu L^2}, \qquad (25)$$

with $m,n=1,2,\ldots$ and $m\geqslant 2n$ [9]. With $\hbar=1$, $\mu=8\pi^2/9$, and side length $L=1$, the quantum energies are $3,7,12,13,19,21,\ldots$. These energies are approached by means of the transfer operator method, by dividing a selected PS Σ into n cells of width $\Delta=\text{length}(\Sigma)/n$. Under the basis,

$$\psi_j(q)=\begin{cases}1/\sqrt{\Delta} & \text{for } q\in j\text{th cell}\\ 0 & \text{otherwise,}\end{cases}$$

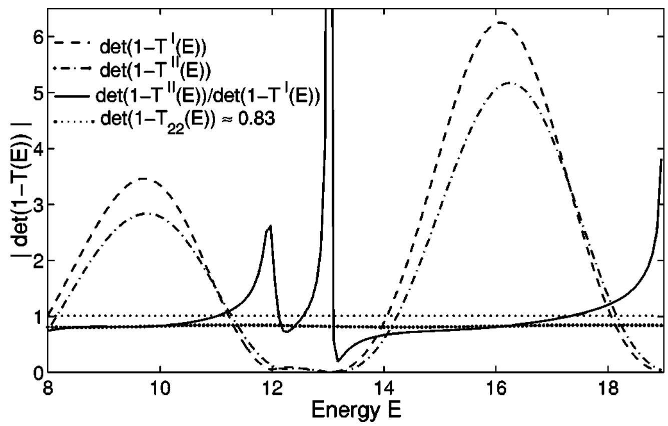

FIG. 3. The peaks of the ratio (26) appear near the exact quantum energies 12, 13, and 19 where the determinant $|\det[1-\mathcal{T}(E)]|$ has minima close to zero, with energy unit $8\hbar^2\pi^2/9\mu L^2$ in Eq. (25).

the transfer operator is discretized into an n-dimensional matrix $\mathbf{T}^{(n)}_{kj}$ with entries

$$\frac{\Delta}{(2\pi i\hbar)^{1/2}}\sum_{\text{class traj}}\sqrt{\left|\frac{\partial^2 S(q_k,q_j',E)}{\partial q_k\partial q_j'}\right|}\times\exp[iS(q_k,q_j',E)/\hbar]-(i\nu\pi/2)$$

where q_j is the center of the jth cell.

For $E=E_{\text{exact}}$, the quantization condition,

$$\det[1-\mathcal{T}(E)]=\lim_{\dim n\to\infty}\det[1-\mathbf{T}^{(n)}(E)]$$

should be zero in principle. However, by the semiclassical approximation, the minima of $|\det[1-\mathcal{T}(E)]|$ are not exactly zero, but only very close to it. Moreover, the locations E of these minima deviate slightly from the exact energies E_{exact}. For the first exact energy, 3, the minima of $|\det[1-\mathcal{T}(E)]|$ on different PS's Σ^{I}, Σ^{II}, and Σ^{III} are located around $E=2.9993$, 3.08, and 3.27, respectively, with 100 cells. For the 298th exact energy, 1008, the minima are located at 1008.18, 1008.47, and 1008.85, respectively, with 500 cells. The best approximation is obtained using Σ^{I} and the worst is obtained using Σ^{III}.

The extent to which formula (19) is satisfied in the approximation is examined by rewriting Eq. (19) as

$$\frac{\det[1-\mathcal{T}^{\mathrm{II}}(E)]}{\det[1-\mathcal{T}^{\mathrm{I}}(E)]}=g(E)=\det[1-\mathcal{T}_{22}(E)]. \qquad (26)$$

For $E\approx E_{\text{exact}}$, two determinants in the ratio are small. A tiny semiclassical error, such as the error of minimum values or locations of these minima, induces a large fluctuation in the ratio of Eq. (26), and yields peaks, as shown in Fig. 3. Apart from these minima, ratio (26) is close to the function $\det[1-\mathcal{T}_{22}(E)]\approx 0.83$, as predicted in formula (19).

In the semiclassical limit $\hbar\to 0$, or equivalently, when E is very large, the deviation in Eq. (26) is expected to decrease. However, if $\det[1-\mathcal{T}^{\mathrm{I}}(E)]$ approaches zero faster than $\det[1-\mathcal{T}^{\mathrm{II}}(E)]$ for large $E\approx E_{\mathrm{exact}}$, then the peaks in Fig. 3 will not disappear as $\hbar\to 0$. Only the width of the peaks decreases. That is, in general, the convergence of the first equality in Eq. (26) in the semiclassical limit $\hbar\to 0$ could be nonuniform.

The determinants on the sections Σ^{III} and Σ^{II} in the triangular billiard B are compared by replacing $\mathcal{T}^{\mathrm{II}}(E)$ in Eq. (26) by $\mathcal{T}^{\mathrm{III}}(E)$, $\mathcal{T}^{\mathrm{I}}(E)$ by $\mathcal{T}^{\mathrm{II}}(E)$, and $\mathcal{T}_{22}(E)$ by $\mathcal{T}_{33}(E)$ to obtain

$$\frac{\det[1-\mathcal{T}^{\mathrm{III}}(E)]}{\det[1-\mathcal{T}^{\mathrm{II}}(E)]}\approx\det[1-\mathcal{T}_{33}(E)]=1. \tag{27}$$

The determinant $\det[1-\mathcal{T}_{33}(E)]$ is 1 because in the triangular billiard, no trajectory runs from $q_3'\in\Sigma_3$ to $q_3\in\Sigma_3$ without passing $\Sigma^{\mathrm{II}}=\Sigma_1\cup\Sigma_2$ and therefore the operator $\mathcal{T}_{33}$ vanishes. In this case, the quantization conditions from Σ^{III} and Σ^{II} are identical.

V. CONCLUSION

According to the theory of Bogomolny's transfer operator method, the zeros of the Fredholm determinant (3) on a Poincaré section of a classical dynamical system are the discrete energies of the corresponding quantum system. This determinant serves as a semiclassical quantization condition. The quantization conditions on different sections are not unique, since the choice of the Poincaré section is arbitrary. However, these conditions can be explicitly related by two formulas (19) and (20), presented in this work. These exact formulas offer an argument as to why different quantization conditions constructed on different classical trajectory segments determine exactly the same energy spectrum of the corresponding quantum system. The formulas presented here are directly reformulated from Bogomolny's quantization condition (3) without further assumption besides the semiclassical approximation.

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